CHAPTER 5
NUMBER SYSTEMS

A number system is a set of symbols or characters which stand for numbers and are used for counting, adding, subtracting, etc. All number systems are related to each other by means of symbols, referred to as digits, although some number systems do not contain all of the same digits of another system. The decimal system will be used as a basis for our discussion of the other number systems.

Two important discoveries have been made since ancient times which greatly simplify the operations of numbers. These are the numeral zero and the principle of place value. The principle of place value consists of giving a numeral a value that depends on its position in the entire number. For example, in the numbers 463, 643, and 364, the 4 has a different value in each by virtue of its position or place value. In the first number it means 4 hundreds, in the second number it means 4 tens, and in the last number it means 4 ones. The zero is used in cases where a number does not have a value for a particular place value; that is, in the number 306 there is no value for tens and the 0 indicates this. We say it is used as a place holder. It would be difficult to express the number three hundred six without using the zero.

We will discuss systems of numbers, basic operations in these systems, and the processes used to convert from one system to another system.

SYSTEMS

Systems of numbers are identified by their radix or base. The base of a number system is a number indicating how many characters or symbols it possesses, including zero. In the Hindu-Arabic system, the system we use and call the decimal system, there are ten symbols or digits. These are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The proper names along with their corresponding bases for several of the different systems are listed as follows:

Base	Name
2	Binary
3	Ternary
4	Quaternary
5	Quinary
6	Senary
7	Septenary
8	Octanary (Octal)
9	Novenary
10	Decimal
12	Duodecimal
16	Sexadecimal (Hexadecimal)
20	Vicenary (Vigesimal)
60	Sexagesimal

DECIMAL SYSTEM

When we count in base ten, or in the decimal system, we begin with 0 and count through 9. When we reach 9 and attempt to count one more unit, we rely on place value; that is, we say 9 and one more is ten. We show this by writing

$$10$$

which in place value notation means one group of ten and no groups of one. While we have a proper name for 10, we could call this "one-zero" in base ten.

The place value chart for base ten is formed by writing the base in columns and then assigning exponents to the base of each column in ascending order from the right to the left starting with zero; that is,

$$\cdots \quad (10)^4 \quad (10)^3 \quad (10)^2 \quad (10)^1 \quad (10)^0$$

The column with value $(10)^0$ is the column to the left of the decimal point. When we write a number in base ten, we do not use a subscript, whereas in other bases we do make the identification of the base by the subscript. In the number 3762, in the place value columns we have

$(10)^3$	$(10)^2$	$(10)^1$	$(10)^0$
3	7	6	2

which means 3 groups of $(10)^3$ or 3000, 7 groups of $(10)^2$ or 700, 6 groups of $(10)^1$ or 60, and 2 groups of $(10)^0$ or 2. Recall that any number raised to the zero power is equal to one. The previous number may be written in polynomial form as

$$3(10)^3 + 7(10)^2 + 6(10)^1 + 2(10)^0$$

In general, then, if B is the base of a system the place value columns are as follows:

$$\cdots B^5 \quad B^4 \quad B^3 \quad B^2 \quad B^1 \quad B^0 \cdot B^{-1} \quad B^{-2} \cdots$$

although we will discuss only the values of a base for B^0 and larger; that is, the whole numbers and not fractions.

QUINARY SYSTEM

When counting in the base five (Quinary) system, we must limit ourselves to the use of only the digits in the system. These digits are 0, 1, 2, 3, and 4. If we start counting with zero, we have

$$0_5, 1_5, 2_5, 3_5, 4_5, 10_5, 11_5, 12_5, 13_5, 14_5, 20_5, \ldots$$

We identify our system by the subscript of 5. Therefore,

$$10_5$$

is really, in place value columns,

...	5^2	5^1	5^0
		1	0

which means one group of 5 and no groups of 1. Counting is shown as

... 5^2	5^1	5^0
		0
		1
		2
		3
		4
	1	0
	1	1
	1	2
	1	3
	1	4
	2	0
	2	1
	2	2
	2	3
	2	4
	3	0

The number 342_5 means 3 groups of 5^2 plus 4 groups of 5^1 plus 2 groups of 5^0. In place value notation it is

5^2	5^1	5^0
3	4	2

BINARY SYSTEM

The only digits in the binary system are 0 and 1. We form the place value columns by starting at the radix, or binary, point (analogous to "decimal point") and assign the base (2) the exponent of 0. Then, moving to the left, we increase the exponent by one for each place value column; that is, each column to the left is a multiple of the one on the right. We write

$$\cdots 2^7 \; 2^6 \; 2^5 \; 2^4 \; 2^3 \; 2^2 \; 2^1 \; 2^0$$

When we count in base 2, we start with zero and proceed as follows:

...	2^2	2^1	2^0
			0
			1
		1	0
		1	1
	1	0	0
	1	0	1
	1	1	0

Notice that the last number counted, that is, 110_2, is one group of 2^2, one group of 2^1, and no groups of one. Written in polynomial form it is

$$1(2)^2 + 1(2)^1 + 0(2)^0$$

In a binary number, each digit has a particular power of the base associated with it. This power of the base is called the positional notation. This is sometimes called the weighting value and depends upon the position of its digit. In the number 101_2 the weighting value of the leftmost digit is $(2)^2$. Notice that the weighting value of any digit is the base raised to a power which is equal to the number of digits to the right of the digit being discussed. That is, in the number 101101_2, the leftmost digit has five digits to its right; therefore, the weighting value of the leftmost digit is $(2)^5$. This weighting value is also called the position coefficient.

OCTAL SYSTEM

In the octal system the digits are 0, 1, 2, 3, 4, 5, 6, and 7. The place value columns, indicated by the weighting values, are

$$\cdots 8^5 \; 8^4 \; 8^3 \; 8^2 \; 8^1 \; 8^0$$

The number 2307_8 means 2 groups of 8^3, 3 groups of 8^2, 0 groups of 8^1, and 7 groups of 8^0.

Counting in base eight is as follows:

0
1
2
3
4
5
6
7
1 0
1 1
1 2
1 3
.
.
.

Notice that after 7_8 we write 10_8 which is read "one-zero" and not "ten."

The number 2307_8 in polynomial form is

$$2(8)^3 + 3(8)^2 + 0(8)^1 + 7(8)^0$$

In general we may express a number as

$$a_1(r)^{n-1} + a_2(r)^{n-2} + \cdots + a_n(r)^{n-n}$$

where a is any digit in the number system, r is the radix or base, and n is the number of digits to the left of the radix or base point. (In the decimal system the radix point is the decimal point.) Again, the number 2307_8 is

$$a_1(r)^{n-1} + a_2(r)^{n-2} + a_3(r)^{n-3} + a_4(r)^{n-4}$$

or

$$2(8)^3 + 3(8)^2 + 0(8)^1 + 7(8)^0$$

where

$$n = 4$$
$$r = 8$$
$$a_1 = 2$$
$$a_2 = 3$$
$$a_3 = 0$$
$$a_4 = 7$$

DUODECIMAL SYSTEM

In the duodecimal (base 12) system we must create two new symbols. These symbols are needed because as we count in base 12; that is,

0
1
2
3
4
5
6
7
8
9

we find we cannot write the next number as 10 because this would indicate one group of twelve and no groups of one and we have not counted that high. We therefore write

:
7
8
9
t
e

where t indicates ten and e indicates eleven in the counting process familiar to us. To count further we write

:
8
9
t
e
1 0
1 1
1 2
:
1 t
1 e
2 0
:

where 12 means one group of $(12)^1$ and two groups of $(12)^0$. The number 1t means one group of $(12)^1$ and ten groups of $(12)^0$. The place value columns for base twelve are:

$$\cdots (12)^4 \quad (12)^3 \quad (12)^2 \quad (12)^1 \quad (12)^0$$

The number $2t9e6_{12}$ is

$$2(12)^4 + t(12)^3 + 9(12)^2 + e(12)^1 + 6(12)^0$$

HEXADECIMAL SYSTEM

This system has a base of sixteen; therefore, we need sixteen different symbols. The symbols we use are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.

The number $9A6F_{16}$ means

$$9(16)^3 + A(16)^2 + 6(16)^1 + F(16)^0$$

In this system A is our familiar ten and F is fifteen. Note that the symbol 10_{16} is not "ten" but instead means "sixteen." The number systems discussed to this point may be compared by use of figure 5-1. In this table assume you are counting objects in each of the systems. Notice that when our count reaches object number eleven in base ten, this object is identified by 1011_2, 21_5, 13_8, e_{12}, and B_{16} in the various bases.

OPERATIONS

The operations we will discuss, for various bases, will be the basic arithmetic operations of addition, subtraction, multiplication, and division. Ease in performing these operations will facilitate ease of understanding conversions from one base to another base which will be discussed later in this chapter.

ADDITION

In general, the rules of arithmetic apply to any number system. Each system has a unique digit addition and digit multiplication table. These tables will be discussed with each system.

Decimal

Addition facts in base ten are shown in figure 5-2. The sign of operation is given in the upper left corner. The addends are indicated by row A and column B. The sums are shown

BASE

OBJECTS COUNTED

10	2	5	8	12	16
0	0	0	0	0	0
1	1	1	1	1	1
2	10	2	2	2	2
3	11	3	3	3	3
4	100	4	4	4	4
5	101	10	5	5	5
6	110	11	6	6	6
7	111	12	7	7	7
8	1000	13	10	8	8
9	1001	14	11	9	9
10	1010	20	12	t	A
11	1011	21	13	e	B
12	1100	22	14	10	C
13	1101	23	15	11	D
14	1110	24	16	12	E
15	1111	30	17	13	F
16	10000	31	20	14	10
17	10001	32	21	15	11
18	10010	33	22	16	12

Figure 5-1.—Number bases.

in the array C. To find the sum C of A + B locate the addends A and B. The sum C will be located where A and B intersect. The commutative principle causes the table to be symmetrical with respect to the diagonal with a negative slope. This is shown by the dotted line.

Quinary

Quinary addition facts are shown in figure 5-3. The table is symmetrical as was the decimal table. In adding 3_5 and 4_5 one should mentally add these and find the sum of 7. There is no symbol of 7 in base five, but one group of five and two groups of one will indicate this sum. That is,

$$\begin{array}{r} 4_5 \\ +\ 3_5 \\ \hline 1\ 2_5 \end{array}$$

where 12_5 is really

5^1	5^0	
1	2	5

In the decimal system a carry of ten is made, but in the quinary system a carry of five is made.

EXAMPLE: Add 324_5 and 433_5.
SOLUTION: Write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline \end{array}$$

Then, 4_5 plus 3_5 is 12_5 therefore, write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 2_5 \end{array}$$

with a carry of one group of five.
Then, 2_5 plus 3_5 is 10_5 and the carry brings the total to 11_5 so we write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 12_5 \end{array}$$

with a carry of one group of $(5)^2$. This gives 3_5 plus 4_5 equals 12_5 and the carry of one brings the total to 13_5. Therefore, we have

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 1312_5 \end{array}$$

EXAMPLE: Add 3042_5 and 4323_5.
SOLUTION: Write

+	0	1	2	3	4	5	6	7	8	9	A
0	0	1	2	3	4	5	6	7	8	9	
1	1	2	3	4	5	6	7	8	9	10	
2	2	3	4	5	6	7	8	9	10	11	
3	3	4	5	6	7	8	9	10	11	12	
4	4	5	6	7	8	9	10	11	12	13	C
5	5	6	7	8	9	10	11	12	13	14	
6	6	7	8	9	10	11	12	13	14	15	
7	7	8	9	10	11	12	13	14	15	16	
8	8	9	10	11	12	13	14	15	16	17	
9	9	10	11	12	13	14	15	16	17	18	
B											

Figure 5-2.—Decimal addition.

+	0	1	2	3	4	A
0	0	1	2	3	4	
1	1	2	3	4	10	
2	2	3	4	10	11	C
3	3	4	10	11	12	
4	4	10	11	12	13	
B						

Figure 5-3.—Quinary addition.

$$\begin{array}{r} 3042_5 \\ +\ 4323_5 \\ \hline \end{array}$$

Then, in steps, with the carry indicated, we have

$$\begin{array}{r} 1 \\ 3042_5 \\ +\ 4323_5 \\ \hline 0_5 \end{array}$$

$$\begin{array}{r} 1 \\ 3042_5 \\ +\ 4323_5 \\ \hline 20_5 \end{array}$$

$$\begin{array}{r} 0 \\ 3042_5 \\ +\ 4323_5 \\ \hline 420_5 \end{array}$$

$$\begin{array}{r} 3042_5 \\ +\ 4323_5 \\ \hline 12420_5 \end{array}$$

This process is quite satisfactory until we try to add, in base five, the following (omit subscripts for simplicity):

$$\begin{array}{r} 44 \\ 43 \\ 24 \\ 34 \\ 24 \\ 33 \\ 23 \\ +\ \ 13 \\ \hline \end{array}$$

We add the 5^0 column and find we have a sum of 28 or 103_5. (103_5 may be found by counting 28 objects.) Our problem is, what do we carry? In this case we carry 10_5 which is 5. Then the next column sum is 26 which is 101_5. Therefore, we write the sum as

$$1013_5$$

PROBLEMS: Add the following base five numbers.

1.
$$\begin{array}{r} 302 \\ +\ 443 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 40321 \\ +\ 43444 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 434 \\ 424 \\ 143 \\ 344 \\ 204 \\ 432 \\ 443 \\ +\ 342 \\ \hline \end{array}$$

ANSWERS:

1. 1300_5
2. 134320_5
3. 110415_5

Binary

Binary addition facts are shown in figure 5-4. Notice that a binary digit has only two possible values, 0 and 1. A carry of two is involved in binary addition. That is, when we add one and one the sum is two, but we have no two so we write 10_2 which indicates one group of $(2)^1$ and no group of one.

EXAMPLE: Add 1011_2 and 1101_2.

SOLUTION: Write

$$\begin{array}{r} 1011_2 \\ +\ 1101_2 \\ \hline \end{array}$$

Then, one and one are two, but "two" is

$$10_2$$

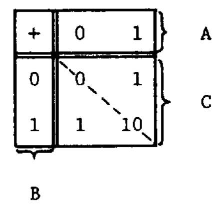

Figure 5-4.—Binary addition.

so we write

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 0 \end{array}$$

and carry a one. The following steps, with the carry indicated, show the completion of our addition.

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 0_2 \end{array}$$

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 00_2 \end{array}$$

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 000_2 \end{array}$$

$$\begin{array}{r} 1011_2 \\ +\ 1101_2 \\ \hline 11000_2 \end{array}$$

Notice in the last step we added three ones which total three, and "three" is written as

$$11_2$$

EXAMPLE: Add 111_2, 101_2, and 11_2.
SOLUTION: Write

$$\begin{array}{r} 1 \\ 111_2 \\ 101_2 \\ +\ 11_2 \\ \hline 1_2 \end{array}$$

$$\begin{array}{r} 1 \\ 111_2 \\ 101_2 \\ +\ 11_2 \\ \hline 11_2 \end{array}$$

$$\begin{array}{r} 111_2 \\ 101_2 \\ +\ 11_2 \\ \hline 1111_2 \end{array}$$

We may verify this by writing

$$\begin{array}{rcr} 111_2 & = & 7 \\ 101_2 & = & 5 \\ +\ 11_2 & = & 3 \\ \hline & & 15 \end{array}$$

and 15 in base two is 1111_2.

PROBLEMS: Add the following base two numbers.

1. $\begin{array}{r} 11 \\ +\ 101 \\ \hline \end{array}$

2. $\begin{array}{r} 101 \\ +\ 1010 \\ \hline \end{array}$

3. $\begin{array}{r} 11 \\ 11 \\ 11 \\ +\ 11 \\ \hline \end{array}$

ANSWERS:

1. 1000_2
2. 1111_2
3. 1100_2

Octal

The octal system has the digits 0, 1, 2, 3, 4, 5, 6, and 7. When an addition carry is made, the carry is eight. The addition facts are shown in figure 5-5.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

A

B

C

Figure 5-5.—Octal addition.

When we add 7_8 and 6_8 we have a sum of thirteen but thirteen in base eight is one group of eight and five groups of one. We write

$$\begin{array}{r} 7_8 \\ +\ 6_8 \\ \hline 15_8 \end{array}$$

EXAMPLE: Add 765_8 and 675_8.
SOLUTION: Write

$$\begin{array}{r} 1 \\ 765_8 \\ +\ 675_8 \\ \hline 2_8 \end{array}$$

$$\begin{array}{r} 1 \\ 765_8 \\ +\ 675_8 \\ \hline 62_8 \end{array}$$

$$\begin{array}{r} 765_8 \\ +\ 675_8 \\ \hline 1662_8 \end{array}$$

PROBLEMS: Add the following octal numbers.

1. $\begin{array}{r} 332 \\ +\ 436 \\ \hline \end{array}$

2. $\begin{array}{r} 703 \\ +\ 677 \\ \hline \end{array}$

3. $\begin{array}{r} 4562 \\ +\ 7541 \\ \hline \end{array}$

ANSWERS:

1. 770_8

2. 1602_8

3. 14323_8

Duodecimal

The addition facts for the base twelve system are shown in figure 5-6. The t equals ten and the e equals eleven. When a carry is made the carry is twelve. When we add 9_{12} and e_{12} we find the sum is twenty and twenty is written as one group of twelve and eight groups of one; that is,

$$\begin{array}{r} 9_{12} \\ +\ e_{12} \\ \hline 18_{12} \end{array}$$

EXAMPLE: Add $8te2_{12}$ and $9e4_{12}$.
SOLUTION: Write

$$\begin{array}{r} 8te2_{12} \\ +\ 9e4_{12} \\ \hline 6_{12} \end{array}$$

$$\begin{array}{r} 1\phantom{_{12}} \\ 8te2_{12} \\ +\ 9e4_{12} \\ \hline t6_{12} \end{array}$$

$$\begin{array}{r} 1\phantom{2_{12}} \\ 8te2_{12} \\ +\ 9e4_{12} \\ \hline 8t6_{12} \end{array}$$

$$\begin{array}{r} 8te2_{12} \\ +\ 9e4_{12} \\ \hline 98t6_{12} \end{array}$$

A

+	0	1	2	3	4	5	6	7	8	9	t	e
0	0	1	2	3	4	5	6	7	8	9	t	e
1	1	2	3	4	5	6	7	8	9	t	e	10
2	2	3	4	5	6	7	8	9	t	e	10	11
3	3	4	5	6	7	8	9	t	e	10	11	12
4	4	5	6	7	8	9	t	e	10	11	12	13
5	5	6	7	8	9	t	e	10	11	12	13	14
6	6	7	8	9	t	e	10	11	12	13	14	15
7	7	8	9	t	e	10	11	12	13	14	15	16
8	8	9	t	e	10	11	12	13	14	15	16	17
9	9	t	e	10	11	12	13	14	15	16	17	18
t	t	e	10	11	12	13	14	15	16	17	18	19
e	e	10	11	12	13	14	15	16	17	18	19	1t

C

B

Figure 5-6.—Duodecimal addition.

PROBLEMS: Add the following duodecimal numbers.

1. $\begin{array}{r} t \\ +\ e \\ \hline \end{array}$

2. $\begin{array}{r} 9t6 \\ +\ e45 \\ \hline \end{array}$

3. $\begin{array}{r} te7t \\ +\ 9979 \\ \hline \end{array}$

ANSWERS:

1. 19_{12}

2. $192e_{12}$

3. 18937_{12}

Hexadecimal

The hexadecimal or base sixteen system addition facts are shown in figure 5-7. There are sixteen symbols needed; therefore, the letters A, B, C, D, E, and F are used for digits greater than 9. In addition in this system groups of sixteen are carried.

EXAMPLE: Add $3A9_{16}$ and $E86_{16}$.

SOLUTION: Write

$$\begin{array}{r} 0 \\ 3A9_{16} \\ +\ E86_{16} \\ \hline F_{16} \end{array}$$

+	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
2	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11
3	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12
4	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13
5	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14
6	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15
7	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16
8	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17
9	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18
A	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19
B	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A
C	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B
D	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C
E	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D
F	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E

Figure 5-7.—Hexadecimal addition.

$$\begin{array}{r} 1 \\ 3A9_{16} \\ +\ E86_{16} \\ \hline 2F_{16} \end{array}$$

$$\begin{array}{r} 3A9_{16} \\ +\ E86_{16} \\ \hline 122F_{16} \end{array}$$

EXAMPLE: Add $BC2_{16}$ and EFA_{16}.
SOLUTION: Write

$$\begin{array}{r} 0 \\ BC2_{16} \\ +\ EFA_{16} \\ \hline C_{16} \end{array}$$

$$\begin{array}{r} 1 \\ BC2_{16} \\ +\ EFA_{16} \\ \hline BC_{16} \end{array}$$

$$\begin{array}{r} BC2_{16} \\ +\ EFA_{16} \\ \hline 1ABC_{16} \end{array}$$

PROBLEMS: Add the following hexadecimal numbers.

1. $\begin{array}{r} 9A6 \\ +\ B84 \\ \hline \end{array}$

2. $\begin{array}{r} ABC \\ +\ EF9 \\ \hline \end{array}$

3. $\begin{array}{r} 87A2 \\ +\ F9EC \\ \hline \end{array}$

ANSWERS:

1. $152A_{16}$
2. $19B5_{16}$
3. $1818E_{16}$

SUBTRACTION

Subtraction in any number system is performed in the same manner as in the decimal system. In the process of addition we were faced with the "carry," and in subtraction we are faced with "borrowing."

Since the process of subtraction is the opposite of addition, we may use the addition tables for subtraction facts for the various bases discussed previously.

Decimal

Figure 5-2 is the addition table for the decimal system. Since this table indicates that

$$A + B = C$$

we may use this table for subtraction facts by writing

$$A + B = C$$

then

$$C - A = B$$

or

$$C - B = A$$

To subtract 8 from 15, find 8 in either the A row or B column. Find where this row or column intersects with a value of 15 for C, then move to the remaining row or column to find the remainder.

This problem, when written in the familiar form of

$$\begin{array}{rl} 15 & \text{minuend} \\ -\ 8 & \text{subtrahend} \\ \hline 7 & \text{remainder} \end{array}$$

requires the use of the "borrow"; that is, when we try to subtract 8 from 5 to obtain a positive remainder, we cannot accomplish this. We borrow the 1 which is really one group of ten. Then, one group of ten and 5 groups of one equals 15 groups of one. Then, 15 groups less 8 groups gives the 7 remainder. While this may seem trivial, it nevertheless points out the process used in all number bases. This process may become confusing, when the base is something other than the familiar decimal base.

Quinary

Figure 5-3 may be used in quinary subtraction in the same manner as figure 5-2 was used in decimal subtraction; that is, 4_5 from 12_5 is 3_5. To find this difference locate 4 in the A row, move down to 12_5 in the C array, then across to 3 in the B column. In the familiar form of

$$\begin{array}{r} 12_5 \\ -\ \ 4_5 \\ \hline 3_5 \end{array}$$

we find we must borrow the 1 from the left place value. This 1 is really one group of 5, therefore one group of 5 added to 2 is 7 and 4 from 7 is 3.

Notice that we solve the previous problem by thinking in base ten but writing in base five. This is permissible because the previous problem may be shown as follows:

$$\begin{aligned} & 12_5 - 4_5 \\ &= 1(5)^1 + 2(5)^0 - 4(5)^0 \\ &= 1(4 + 1) + 2(1) - 4(1) \\ &= 4 + 1 + 2 - 4 \\ &= 4 - 4 + 1 + 2 \\ &= 0 + 1 + 2 \\ &= 3 \end{aligned}$$

where all numbers are in base five.

EXAMPLE: Subtract 43_5 from 431_5.

SOLUTION: Write

$$\begin{array}{r} 431_5 \\ -\ \ 43_5 \\ \hline \end{array}$$

Thinking in decimal notation, we borrow 1 group of five from the 3 groups of five and write

$$\begin{array}{r} 25 \\ 4\cancel{3}1_5 \\ -\ \ 43_5 \\ \hline 3_5 \end{array}$$

and then borrow 1 group of $(5)^2$ from the 4 groups of $(5)^2$ which gives

$$\begin{array}{r} 355 \\ \cancel{4}21_5 \\ -\ \ 43_5 \\ \hline 3_5 \end{array}$$

and we write

$$\begin{array}{r} 55 \\ 321_5 \\ -\ \ 43_5 \\ \hline 333_5 \end{array}$$

Notice that in the indicated "borrow" we write the numeral which indicates the base. This is for explanation purposes only; there is really no numeral 5 in base five. This holds for the following indicated "borrowing" process in the other bases.

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 23_5 \\ -\ \ 4_5 \\ \hline \end{array}$

2. $\begin{array}{r} 432_5 \\ -\ 344_5 \\ \hline \end{array}$

3. $\begin{array}{r} 4032_5 \\ -\ \ 343_5 \\ \hline \end{array}$

ANSWERS:

1. 14_5
2. 33_5
3. 3134_5

Binary

When subtracting in base two, the addition table in figure 5-4 is used. To subtract 1_2 from 10_2 the borrow of two is used. That is,

$$\begin{array}{r} 10_2 \\ -\ 1_2 \\ \hline \end{array}$$

is one group of $(2)^1$ and no group of $(2)^0$ minus one group of $(2)^0$. Thinking in base ten, this is 2 minus 1 which is 1. This may be verified by using figure 5-4.

EXAMPLE: Subtract 11_2 from 101_2.
SOLUTION: Write

$$\begin{array}{r} 101_2 \\ -\ 11_2 \\ \hline \end{array}$$

Then, 1 from 1 is 0 and write

$$\begin{array}{r} 101_2 \\ -\ 11_2 \\ \hline 0_2 \end{array}$$

Now, borrow the left hand 1 which has the value two when moved to the next column to the right. 1 from 2 is 1, and

$$\begin{array}{r} 2 \\ \cancel{1}01_2 \\ -\ 11_2 \\ \hline 10_2 \end{array}$$

PROBLEMS: Perform the indicated operation in the following.

1. $\begin{array}{r} 11_2 \\ -\ 1_2 \\ \hline \end{array}$

2. $\begin{array}{r} 1011_2 \\ -\ 101_2 \\ \hline \end{array}$

3. $\begin{array}{r} 1000_2 \\ -\ 101_2 \\ \hline \end{array}$

ANSWERS:

1. 10_2
2. 110_2
3. 11_2

Octal

Figure 5-5 contains the octal subtraction facts in that

$$C - B = A$$

or

$$C - A = B$$

EXAMPLE: Find the remainder when 6_8 is subtracted from 13_8.
SOLUTION: If, in figure 5-5,

$$C = 13_8$$

and

$$B = 6_8$$

then

$$C - B = A$$

$$13_8 - 6_8 = 5_8$$

EXAMPLE: Subtract 326_8 from 432_8.
SOLUTION: Write

$$\begin{array}{r} 432_8 \\ -\ 326_8 \\ \hline \end{array}$$

then, borrow one group of eight from the 3 which gives eight plus two equals ten. Six from ten is four. Write

$$\begin{array}{r} 28 \\ 4\cancel{3}2_8 \\ -\ 326_8 \\ \hline 4_8 \end{array}$$

and then

$$\begin{array}{r} 8 \\ 422_8 \\ -\ 326_8 \\ \hline 104_8 \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 637_8 \\ -\ 226_8 \\ \hline \end{array}$

2. $\begin{array}{r} 3206_8 \\ -\ 2737_8 \\ \hline \end{array}$

3. $\begin{array}{r} 4006_8 \\ -\ 1767_8 \\ \hline \end{array}$

ANSWERS:

1. 411_8
2. 247_8
3. 2017_8

Duodecimal

Through the use of figure 5-6 we find that 13_{12} minus 9_{12} is 6_{12}. This may be explained by writing

$$\begin{array}{r} 13_{12} \\ -\ 9_{12} \\ \hline \end{array}$$

We borrow one group of twelve and add it to the three groups of one to obtain fifteen. Then, nine from fifteen is six. Therefore,

$$13_{12} - 9_{12} = 6_{12}$$

Here, as before, we think in base ten and write in the base being used.

EXAMPLE: Subtract $2e9_{12}$ from $t64_{12}$.
SOLUTION: Write

$$\begin{array}{r} t64_{12} \\ -\ 2e9_{12} \\ \hline \end{array}$$

Borrow one group of twelve and add it to four to obtain sixteen. Then nine from sixteen is seven. Write

$$\begin{array}{rccc} & & 5 & 12 \\ & t & \cancel{6} & 4_{12} \\ - & 2 & e & 9_{12} \\ \hline & & & 7_{12} \end{array}$$

Then, borrow one group from t, the $(12)^2$ column, and add it to the five groups of $(12)^1$ to obtain seventeen groups of $(12)^1$ minus e groups of $(12)^1$ for a remainder of six groups of $(12)^1$. Write

$$\begin{array}{rccc} & 9 & 12 & 12 \\ & \cancel{t} & 5 & 4_{12} \\ - & 2 & e & 9_{12} \\ \hline & & 6 & 7_{12} \end{array}$$

then, 2_{12} from 9_{12} is 7_{12}, therefore,

$$\begin{array}{rccc} & 9 & 12 & 12 \\ & \cancel{t} & 5 & 4_{12} \\ - & 2 & e & 9_{12} \\ \hline & 7 & 6 & 7_{12} \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 96e_{12} \\ -\ 25t_{12} \\ \hline \end{array}$

2. $\begin{array}{r} 6t9_{12} \\ -\ 37e_{12} \\ \hline \end{array}$

3. $\begin{array}{r} e76_{12} \\ -\ 9te_{12} \\ \hline \end{array}$

ANSWERS:

1. 711_{12}
2. $32t_{12}$
3. 187_{12}

Hexadecimal

In figure 5-7 we use the symbols α (Alpha), β (Beta), and γ (Gamma) in place of A, B, and C for the row, column, and array, because A, B, and C are used as symbols in the hexadecimal system. If

$$\alpha + \beta = \gamma$$

then

$$\gamma - \beta = \alpha$$

and

$$\gamma - \alpha = \beta$$

Subtraction in this system is the same as in the other systems previously discussed except a borrow of sixteen is made when required.

EXAMPLE: Find the remainder when $39E_{16}$ is subtracted from $9C6_{16}$.

SOLUTION: Write

$$\begin{array}{r} 9C6_{16} \\ -\ 39E_{16} \\ \hline \end{array}$$

Then, borrow one group of sixteen from C and add it to six to obtain twenty-two. E (fourteen) from twenty-two is eight. Write

$$\begin{array}{rccc} & & B & 16 \\ & 9 & \cancel{C} & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & & & 8_{16} \end{array}$$

Then, nine from B (eleven) is two, therefore

$$\begin{array}{rccc} & & & 16 \\ & 9 & B & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & & 2 & 8_{16} \end{array}$$

and nine minus three is six, then

$$\begin{array}{rccc} & & & 16 \\ & 9 & B & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & 6 & 2 & 8_{16} \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 6F3_{16} \\ -\ 26A_{16} \\ \hline \end{array}$

2. $\begin{array}{r} 70C_{16} \\ -\ 3FD_{16} \\ \hline \end{array}$

3. $\begin{array}{r} DEF_{16} \\ -\ ABC_{16} \\ \hline \end{array}$

ANSWERS:

1. 489_{16}
2. $30F_{16}$
3. 333_{16}

Subtraction By Complements

Digital computers are generally unable to perform subtraction in the manner previously discussed because the process of borrowing is inconvenient and expensive to mechanize. Therefore, the process of addition of complements is used in place of subtraction. By complement we mean the number or quantity required to fill or complete something in respect to a known reference.

The nines complement of a decimal number is that number which, when added to an original number, will yield nines in each place value column of the original number; that is, the nines complement of 32 is 67 because when 67 is added to 32 the sum is 99.

If we now add one to the nines complement of 32, that is,

$$\begin{array}{r} 67 \\ +\ 1 \\ \hline 68 \end{array}$$

we have the tens complement of 32. Notice that if we add a number (32) and its tens complement (68) we have a sum of 100. Therefore, we define the tens complement of a number as that number which, when added to the original number, yields a 1 in the next higher place value column than the highest contained in the original number. This 1 is followed by zeros in all other place value columns. The tens complement of 39 is

$$100 - 39 = 61$$

Notice that the tens complement is in reference to a power of ten equal to the number of place value columns in the original number. This may be shown as follows:

Number	Tens Complement	Reference
8	2	10^1
36	64	10^2
704	296	10^3

Rather than subtract the subtrahend from the minuend we may add the tens complement of the subtrahend (found with reference to the power of ten one place value higher than either the subtrahend or minuend) to the minuend and then decrease this sum by the reference power of ten used. A step-by-step process is shown to explain the preceding statement.

EXAMPLE: Subtract 26 from 49 using complements.

SOLUTION: Write

$$\begin{array}{r} 49 \\ -\ 26 \\ \hline \end{array}$$

The complement of 26 is 74:

$$100 - 26 = 74$$

This may be rearranged as

$$100 - 74 = 26$$

Now, instead of writing

$$49 - 26$$

write

$$\begin{aligned} & 49 - (100 - 74) \\ = {} & 49 - 100 + 74 \\ = {} & 49 + 74 - 100 \\ = {} & 23 \end{aligned}$$

This indicates that the minuend (49) plus the tens complement (74) of the subtrahend are added, then the reference power of ten used (100) is subtracted to give the difference of (23). Notice that we could write

$$\begin{array}{r} 49 \\ -\ 26 \\ \hline \end{array}$$

equals

$$\begin{array}{l} \quad 49 \\ +\ \ 74 - 100 \\ \hline \ \ 123 - 100 \\ \\ = 23 \end{array}$$

We had only to drop the digit (1) found in the left place value column higher than the highest place value column of the subtrahend. Generally, this digit (1), which is developed, indicates that the difference (23) is a positive value.

EXAMPLE: Subtract 36 from 429 using complements.

SOLUTION: Write

$$\begin{array}{rl} 429 & \text{minuend} \\ -\ \ 36 & \text{subtrahend} \\ \hline \end{array}$$

then

$$\begin{array}{rl} 429 & \text{minuend} \\ +\ 964 & \text{tens complement of subtrahend} \\ & \text{with reference to } 10^3 \\ \hline 1393 & \end{array}$$

Now, rather than drop the 1 in 1393 change it to + which indicates a positive answer of 393.

The procedure for subtracting a larger number from a smaller number is slightly different from the previous example. The 1 developed previously will not be developed and a zero will replace the 1. The zero indicates a negative value but the apparent difference is the complement of the true remainder. This is shown in the following example.

EXAMPLE: Subtract 362 from 127.

SOLUTION: Write

$$\begin{array}{rl} 127 & \text{minuend} \\ -\ 362 & \text{subtrahend} \\ \hline \end{array}$$

then

127	minuend
+ 638	tens complement of subtrahend
0765	apparent remainder

Since there was no 1 generated, we find the true remainder by taking the complement of the apparent remainder and also replacing the 0 with a negative sign to indicate a negative value.

The process of subtraction by using complements in binary is similar to that of decimal. The ones complement of a binary number is found by replacing all ones by zeros and all zeros by ones. This process is called inversion. Therefore, the ones complement of 1001_2 is 0110_2. The twos complement is found by adding a 1 to the ones complement; that is,

Number	Ones Complement	Twos Complement
101_2	010_2	011_2
1011_2	0100_2	0101_2
1111_2	0000_2	0001_2

Notice that the number 101_2 plus its twos complement 011_2 equals 1000_2 which has a 1 developed in the next higher place value column and followed by zeros. The same technique is followed in binary as was used in decimal.

EXAMPLE: Subtract 01101_2 from 11001_2 using complements.

SOLUTION: Write

11001_2	minuend
$-\ 01101_2$	subtrahend

then

11001_2	minuend
$+\ 10011_2$	twos complement of subtrahend
101100_2	difference
= + 01100	positive difference

As in the decimal system, the 1 which is developed indicates a positive answer. If a zero is present, it indicates a negative apparent answer and the true answer is the complement of the apparent answer.

EXAMPLE: Subtract 1011_2 from 1001_2 using complements.

SOLUTION: Write

1001_2	minuend
$-\ 1011_2$	subtrahend

then

1001_2	minuend
$+\ 0101_2$	twos complement of subtrahend
01110_2	apparent difference
$= -\ 0010_2$	twos complement of the apparent difference with the zero indicating a negative difference

CAUTION: When finding the twos complement of a number, do not forget to add a 1 after the inversion process.

PROBLEMS: Subtract in binary using the complements.

1. 1101 from 1110
2. 101 from 1011
3. 1101 from 1001
4. 111 from 10

ANSWERS:

1. $+\ 0001_2$
2. $+\ 0110_2$
3. $-\ 0100_2$
4. $-\ 101_2$

MULTIPLICATION

Multiplication in any number system is performed in the same manner as in the decimal system. Each system has a unique digit multiplication table. These tables will be discussed with each system. The rows, columns, and arrays of these tables are labeled in the same fashion as the addition tables. Only the sign of operation and array values are different.

x	0	1	2	3	4	5	6	7	8	9	A
0	0	0	0	0	0	0	0	0	0	0	
1	0	1	2	3	4	5	6	7	8	9	
2	0	2	4	6	8	10	12	14	16	18	
3	0	3	6	9	12	15	18	21	24	27	
4	0	4	8	12	16	20	24	28	32	36	C
5	0	5	10	15	20	25	30	35	40	45	
6	0	6	12	18	24	30	36	42	48	54	
7	0	7	14	21	28	35	42	49	56	63	
8	0	8	16	24	32	40	48	56	64	72	
9	0	9	18	27	36	45	54	63	72	81	
B											

Figure 5-8.—Decimal multiplication.

Decimal

In multiplication in the decimal system, certain rules are followed which use the decimal digit multiplication and decimal digit addition tables. These rules are well known and apply to direct multiplication in any number system. Figure 5-8 shows the decimal multiplication facts.

The direct method of multiplication of decimal numbers is shown in the following example.

EXAMPLE: Multiply 32 by 25.

SOLUTION: Write

$$25 = 20 + 5$$

then

$$\begin{aligned} &32(25) \\ &= 32(20 + 5) \\ &= 32(20) + 32(5) \\ &= 640 + 160 \\ &= 800 \end{aligned}$$

The same problem written as

$$\begin{array}{r} 32 \\ \underline{\times\ 25} \end{array}$$

gives

$$\begin{aligned} &32(5) \\ &= 160 \quad \text{partial product} \end{aligned}$$

then

$$\begin{aligned} &32(20) \\ &= 640 \quad \text{partial product} \end{aligned}$$

then

$$\begin{aligned} &160 + 640 \\ &= 800 \quad \text{product} \end{aligned}$$

The technique generally used is

$$\begin{array}{r} 32 \\ \underline{\times\ 25} \\ 160 \\ \underline{64\ } \\ 800 \end{array}$$

Notice that the 64 really represents 640 but the zero is omitted.

EXAMPLE: Multiply 306 by 762.

SOLUTION: Write

$$\begin{array}{rl} 306 & \text{factor} \\ \underline{\times\ 762} & \text{factor} \\ 612 & \text{partial product} \\ 1836\ & \text{partial product} \\ \underline{2142\ \ } & \text{partial product} \\ 233172 & \text{product} \end{array}$$

Quinary

Figure 5-9 shows the multiplication facts for base five. Notice that when we multiply 3_5 by 2_5 we think in base ten and write the product in base five; that is, $3_5 \times 2_5$ is six, and six in base five is one group of $(5)^1$ and one group of $(5)^0$. Therefore, we write

$$\begin{array}{r} 3_5 \\ \underline{\times\ 2_5} \\ 11_5 \end{array}$$

x	0	1	2	3	4	A
0	0	0	0	0	0	
1	0	1	2	3	4	
2	0	2	4	11	13	C
3	0	3	11	14	22	
4	0	4	13	22	31	
B						

Figure 5-9.—Quinary multiplication.

EXAMPLE: Multiply 304_5 by 24_5.
SOLUTION: Write

$$\begin{array}{r} 304_5 \\ \underline{\times\ 24_5} \end{array}$$

Then, four times four is sixteen and sixteen in base five is 31_5. (See fig. 5-9.) Therefore, we write

$$\begin{array}{r} 304_5 \\ \underline{\times\ 24_5} \\ 1_5 \end{array}$$

Now, four times zero is zero and the carry of three gives three. Therefore,

$$\begin{array}{r} 304_5 \\ \underline{\times\ 24_5} \\ 31_5 \end{array}$$

and $3_5 \times 4_5$ gives 22_5. (See fig. 5-9.) The first partial product is

$$\begin{array}{rl} 304_5 & \\ \underline{\times\ 24_5} & \\ 2231_5 & \text{partial product} \end{array}$$

This same procedure is used to find the second partial product as

$$\begin{array}{r} 304_5 \\ \underline{\times\ 24_5} \\ 2231_5 \\ 1113_5 \end{array}$$

Then, adding the partial products we find

$$\begin{array}{rl} 304_5 & \\ \underline{\times\ 24_5} & \\ 2231_5 & \\ \underline{1113_5} & \\ 13411_5 & \text{product} \end{array}$$

PROBLEMS: Multiply the following:

1. $\begin{array}{r} 23_5 \\ \underline{\times\ 41_5} \end{array}$

2. $\begin{array}{r} 3004_5 \\ \underline{\times\ 321_5} \end{array}$

3. $\begin{array}{r} 4342_5 \\ \underline{\times\ 434_5} \end{array}$

ANSWERS:

1. 2043_5
2. 2020334_5
3. 4233133_5

Binary

Figure 5-10 shows the multiplication facts for the binary system. This is the most simple set of facts of any of the number systems and as will be seen the only difficulty in binary multiplication may be in the addition of the partial products.

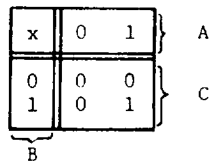

Figure 5-10.—Binary multiplication.

EXAMPLE: Multiply 101_2 by 1101_2.
SOLUTION: Write

$$\begin{array}{r} 1101_2 \\ \times\ 101_2 \\ \hline \end{array}$$

The partial products and the products are as follows:

$$\begin{array}{rl} 1101_2 & \\ \times\ 101_2 & \\ \hline 1101 & \text{partial product} \\ 11010 & \text{partial product} \\ \hline 1000001 & \text{product} \end{array}$$

As in the addition section, the problem that may be encountered in the addition of the partial products is what to carry. The following example will illustrate this problem.
EXAMPLE: Multiply 1111_2 by 111_2
SOLUTION: Write

$$\begin{array}{r} 1111_2 \\ \times\ 111_2 \\ \hline 1111 \\ 1111 \\ 1111 \end{array}$$

We add the partial products by writing

$$\begin{array}{r} 1111_2 \\ \times\ 111_2 \\ \hline 1111 \\ 1111 \\ 1111 \\ \hline 01 \end{array}$$

and when we add the four ones we find four is written in binary as 100_2. We write the zero, then we must carry the 10_2. The symbol 10_2 is really two, thinking in base ten; therefore, we carry two and when two is added to the next three ones we have five. Five is written as 101_2; therefore, we write 1 and carry the 10_2 or two. Two and two are four so we write zero and carry 10_2 or two. Finally, two and one are three and we write 11_2. The entire addition process is shown as follows:

$$\begin{array}{r} 1111 \\ 1111 \\ 1111 \\ \hline 1101001_2 \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 1010_2 \\ \times\ 101_2 \\ \hline \end{array}$

2. $\begin{array}{r} 1110_2 \\ \times\ 111_2 \\ \hline \end{array}$

3. $\begin{array}{r} 11011_2 \\ \times\ 1101_2 \\ \hline \end{array}$

ANSWERS:

1. 110010_2
2. 1100010_2
3. 101011111_2

Octal

Base eight multiplication facts are given in figure 5-11. When multiplying 6_8 by 7_8 we find the product by thinking "six times seven is forty-two" and writing forty-two as five groups of eight and two groups of one or

$$6_8 \times 7_8 = 52_8$$

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Figure 5-11.—Octal multiplication.

EXAMPLE: Multiply 41_8 by 23_8.
SOLUTION: Write

$$\begin{array}{r} 41_8 \\ \times\ 23_8 \\ \hline 143 \\ 102 \\ \hline 1163_8 \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 703_8 \\ \times\ 24_8 \\ \hline \end{array}$

2. $\begin{array}{r} 324_8 \\ \times\ 103_8 \\ \hline \end{array}$

3. $\begin{array}{r} 762_8 \\ \times\ 765_8 \\ \hline \end{array}$

ANSWERS:

1. 21474_8
2. 33574_8
3. 747232_8

Duodecimal

Multiplication facts for the base twelve system are shown in figure 5-12. The process of multiplication is the same as in other bases.
EXAMPLE: Multiply 9_{12} by 5_{12}.
SOLUTION: Write

$$\begin{array}{r} 9_{12} \\ \times\ 5_{12} \\ \hline \end{array}$$

x	0	1	2	3	4	5	6	7	8	9	t	e
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	t	e
2	0	2	4	6	8	t	10	12	14	16	18	1t
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	t	13	18	21	26	2e	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2e	36	41	48	53	5t	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
t	0	t	18	26	34	42	50	5t	68	76	84	92
e	0	e	1t	29	38	47	56	65	74	83	92	t1

Figure 5-12.—Duodecimal multiplication.

Nine times five is forty-five in base ten, and forty-five is written as three groups of twelve and nine groups of one, in base twelve; that is,

$$\begin{array}{r} 9_{12} \\ \times\ 5_{12} \\ \hline 39_{12} \end{array}$$

EXAMPLE: Multiply 5_{12} by 7_{12}.
SOLUTION: Write

$$\begin{array}{r} 5_{12} \\ \times\ 7_{12} \\ \hline \end{array}$$

and thirty-five in base ten is written as two groups of twelve and eleven groups of one, in base twelve; therefore,

$$\begin{array}{r} 5_{12} \\ \times\ 7_{12} \\ \hline 2e_{12} \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 29_{12} \\ \times\ 32_{12} \\ \hline \end{array}$

2. $\begin{array}{r} t7_{12} \\ \times\ 31_{12} \\ \hline \end{array}$

3. $\begin{array}{r} te6_{12} \\ \times\ e_{12} \\ \hline \end{array}$

ANSWERS:

1. 886_{12}
2. 2877_{12}
3. $t066_{12}$

Hexadecimal

Figure 5-13 gives the multiplication facts for base sixteen and figure 5-7 gives the addition facts. By use of both of these tables, the following examples and problems become self-explanatory.

EXAMPLE: Find the product of $6C_{16}$ and 98_{16}.
SOLUTION: Write

$$\begin{array}{r} 6C_{16} \\ \times\ 98_{16} \\ \hline 360 \\ 3CC \\ \hline 4020_{16} \end{array}$$

EXAMPLE: Find the product of $3A7_{16}$ and 84_{16}.
SOLUTION: Write

$$\begin{array}{r} 3A7_{16} \\ \times\ 84_{16} \\ \hline E9C \\ 1D38 \\ \hline 1E21C_{16} \end{array}$$

PROBLEMS: Find the product of the following.

1. $\begin{array}{r} 67_{16} \\ \times\ 31_{16} \\ \hline \end{array}$

2. $\begin{array}{r} ABC_{16} \\ \times\ 32_{16} \\ \hline \end{array}$

3. $\begin{array}{r} 678_{16} \\ \times\ 302_{16} \\ \hline \end{array}$

ANSWERS:

1. $13B7_{16}$
2. $218B8_{16}$
3. $1374F0_{16}$

DIVISION

The process of division is the opposite of multiplication; therefore, we may use the multiplication tables for the various bases to show division facts. We will define division by writing

x	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	α
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	
2	0	2	4	6	8	A	C	E	10	12	14	16	18	1A	1C	1E	
3	0	3	6	9	C	F	12	15	18	1B	1E	21	24	27	2A	2D	
4	0	4	8	C	10	14	18	1C	20	24	28	2C	30	34	38	3C	
5	0	5	A	F	14	19	1E	23	28	2D	32	37	3C	41	46	4B	
6	0	6	C	12	18	1E	24	2A	30	36	3C	42	48	4E	54	5A	γ
7	0	7	E	15	1C	23	2A	31	38	3F	46	4D	54	5B	62	69	
8	0	8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	
9	0	9	12	1B	24	2D	36	3F	48	51	5A	63	6C	75	7E	87	
A	0	A	14	1E	28	32	3C	46	50	5A	64	6E	78	82	8C	96	
B	0	B	16	21	2C	37	42	4D	58	63	6E	79	84	8F	9A	A5	
C	0	C	18	24	30	3C	48	54	60	6C	78	84	90	9C	A8	B4	
D	0	D	1A	27	34	41	4E	5B	68	75	82	8F	9C	A9	B6	C3	
E	0	E	1C	2A	38	46	54	62	70	7E	8C	9A	A8	B6	C4	D2	
F	0	F	1E	2D	3C	4B	5A	69	78	87	96	A5	B4	C3	D2	E1	

β

Figure 5-13.—Hexadecimal multiplication.

$$\frac{C}{B} = A \text{ if, and only if, } AB = C,\ B \neq 0$$

We show this by use of figure 5-8. That is, if

$$C = 42$$

and

$$B = 7$$

then

$$A = 6$$

Notice that the value of C is the intersection of the values of A and B.

For the remainder of this section on division we will use examples and problems for the various number bases along with their respective multiplication tables.

Decimal

EXAMPLE: Divide 54 by 9.

$$\begin{array}{r} 6 \\ 9\overline{)54} \\ \underline{54} \\ 0 \end{array}$$

EXAMPLE: Divide 252 by 6.
SOLUTION: Write

$$\begin{array}{r} 42 \\ 6\overline{)252} \\ \underline{24} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

Quinary

EXAMPLE: Divide 22_5 by 4_5.
SOLUTION: Write

$$\begin{array}{r} 3_5 \\ 4\overline{)22} \\ \underline{22} \\ 0 \end{array}$$

EXAMPLE: Divide 2013_5 by 3_5.
SOLUTION: Write

$$\begin{array}{r} 321_5 \\ 3\overline{)2013} \\ \underline{14} \\ 11 \\ \underline{11} \\ 03 \\ \underline{3} \\ 0 \end{array}$$

PROBLEMS: Divide the following.

1. 134_5 by 4_5
2. 2231_5 by 4_5
3. 2131_5 by 3_5

ANSWERS:

1. 21_5
2. 304_5
3. 342_5

Octal

EXAMPLE: Divide 234_8 by 6_8.
SOLUTION: Write

$$\begin{array}{r} 32_8 \\ 6\overline{)234} \\ \underline{22} \\ 14 \\ \underline{14} \\ 0 \end{array}$$

EXAMPLE: Divide 765_8 by 4_8.
SOLUTION: Write

$$\begin{array}{r} 175_8 \\ 4\overline{)765} \\ \underline{4} \\ 36 \\ \underline{34} \\ 25 \\ \underline{24} \\ 1 \quad \text{remainder} \end{array}$$

PROBLEMS: Divide the following.

1. 202_8 by 5_8
2. 1634_8 by 7_8
3. 372_8 by 12_8

ANSWERS:

1. 32_8
2. 204_8
3. 31_8

Binary

EXAMPLE: Divide 1111_2 by 11_2.
SOLUTION: Write

$$\begin{array}{r} 101_2 \\ 11\overline{)1111} \\ \underline{11} \\ 011 \\ \underline{11} \\ 0 \end{array}$$

EXAMPLE: Divide 101_2 by 10_2.
SOLUTION: Write

$$\begin{array}{rl} 10_2 & \\ 10\overline{)101} & \\ \underline{10} & \\ 1 & \text{remainder} \end{array}$$

PROBLEMS: Divide the following

1. 10110_2 by 10_2
2. 1000001_2 by 101_2
3. 100000_2 by 100_2

ANSWERS:

1. 1011_2
2. 1101_2
3. 1000_2

Duodecimal

EXAMPLE: Divide 446_{12} by 6_{12}.
SOLUTION: Write

$$\begin{array}{r} 89_{12} \\ 6\overline{)446} \\ \underline{40} \\ 46 \\ \underline{46} \\ 0 \end{array}$$

EXAMPLE: Divide 417_{12} by 5_{12}.
SOLUTION: Write

$$\begin{array}{r} 9e_{12} \\ 5\overline{)417} \\ \underline{39} \\ 47 \\ \underline{47} \\ 0 \end{array}$$

PROBLEMS: Divide the following.

1. $2e4_{12}$ by 4_{12}
2. $1e23_{12}$ by 3_{12}
3. $19t_{12}$ by 2_{12}

ANSWERS:

1. $8t_{12}$
2. 789_{12}
3. te_{12}

Hexadecimal

EXAMPLE: Divide $D4E_{16}$ by 2_{16}.
SOLUTION: Write

$$\begin{array}{r} 6A7_{16} \\ 2\overline{)D4E} \\ \underline{C} \\ 14 \\ \underline{14} \\ E \\ \underline{E} \\ 0 \end{array}$$

EXAMPLE: Divide $13BC_{16}$ by 3_{16}.
SOLUTION: Write

$$\begin{array}{r} 694_{16} \\ 3\overline{)13BC} \\ \underline{12} \\ 1B \\ \underline{1B} \\ C \\ \underline{C} \\ 0 \end{array}$$

PROBLEMS: Divide the following

1. $27B5_{16}$ by 5_{16}
2. $4B24_{16}$ by 7_{16}
3. $2CCE_{16}$ by A_{16}

ANSWERS:

1. $7F1_{16}$
2. ABC_{16}
3. $47B_{16}$

CONVERSIONS

It has been shown that place value is the determining factor in evaluating a number. We will make extensive use of this idea in discussing the various methods of conversions.

NON-DECIMAL TO DECIMAL

In order to convert a non-decimal number to a decimal number we make use of the polynomial form. That is, we write the non-decimal number in polynomial form and then carry out the indicated operations.

EXAMPLE: Convert 634_8 to decimal.
SOLUTION: Write

$$\begin{aligned} 634_8 &= 6(8)^2 + 3(8)^1 + 4(8)^0 \\ &= 6(64) + 3(8) + 4(1) \\ &= 384 + 24 + 4 \\ &= 412 \end{aligned}$$

EXAMPLE: Convert $7t0e_{12}$ to decimal; that is, if

$$7t0e_{12} = X_{10}, \text{ then } X_{10} = ?$$

SOLUTION: Write

$$\begin{aligned} 7t0e_{12} &= 7(12)^3 + 10(12)^2 + 0(12)^1 + 11(12)^0 \\ &= 7(1728) + 10(144) + 0(12) + 11(1) \\ &= 12096 + 1440 + 0 + 11 \\ &= 13547 \end{aligned}$$

Therefore, $X_{10} = 13547$.

Another method of non-decimal to decimal conversion is by synthetic substitution. This method is shown in the following example.

EXAMPLE: Convert 634_8 to decimal.
SOLUTION: Write

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ \hline \end{array}$$

Bring down the six

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ \hline & 6 & & \end{array}$$

Multiply the six by the base (expressed in decimal form) and carry the decimal product to the next lower place value column.

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & \\ \hline & 6 & & \end{array}$$

Add the three and the carried product

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & \\ \hline & 6 & 51 & \end{array}$$

Multiply this sum by the base and carry to the next lower place value column.

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & 408 \\ \hline & 6 & 51 & \end{array}$$

Add the four and the carried product to find the decimal equivalent of 634_8 to be

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & 408 \\ \hline & 6 & 51 & 412 \end{array}$$

and 412 is the decimal equivalent of 634_8. The entire previous process may be shown, without carrying out the multiplication, as

$$\begin{array}{c|ccc} 8 & 6 & 3 & 4 \\ \hline & 6 & & \end{array}$$

$$\begin{array}{c|ccc} 8 & 6 & 3 & 4 \\ & & 6(8) & \\ \hline & 6 & & \end{array}$$

$$\begin{array}{c|ccc} 8 & 6 & 3 & 4 \\ & & 6(8) & \\ \hline & 6 & 6(8)+3 & \end{array}$$

$$\begin{array}{c|ccc} 8 & 6 & 3 & 4 \\ & & 6(8) & 8[6(8)+3] \\ \hline & & 6(8)+3 & \end{array}$$

$$\begin{array}{c|ccc} 8 & 6 & 3 & 4 \\ & & 6(8) & 8[6(8)+3] \\ \hline & 6 & 6(8)+3 & 8[6(8)+3]+4 \end{array}$$

and

$$8[6(8)+3] + 4$$

is

$$6(8)^2 + 3(8) + 4$$

which is really 634_8 written in polynomial form.

EXAMPLE: Convert $7te_{12}$ to decimal.

SOLUTION: Write

$$\begin{array}{c|ccc} 12 & 7 & 10 & 11 \\ & & 7(12) & 12[7(12)+10] \\ \hline & 7 & 7(12)+10 & 12[7(12)+10]+11 \end{array}$$

and

$$\begin{aligned} & 12[7(12) + 10] + 11 \\ = \; & 7(12)^2 + 10(12)^1 + 11(12)^0 \\ = \; & 1008 + 120 + 11 \\ = \; & 1139 \end{aligned}$$

A third method of converting a number from non-decimal to decimal is by use of repeated division where the remainders indicate the decimal equivalent. The divisor is ten expressed in the non-decimal number.

EXAMPLE: Convert 634_8 to decimal.

SOLUTION: Ten expressed in base eight is 12; therefore, write

$$12\overline{)634}$$

This division is carried out in base eight.

$$\begin{array}{r} 51 \\ 12\overline{)634} \\ \underline{62} \\ 14 \\ \underline{12} \\ R_1 = 2_8 = 2 \end{array}$$

The quotient is now divided by 12_8.

$$\begin{array}{r} 4 \\ 12\overline{)51} \\ \underline{50} \\ R_2 = 1_8 = 1 \end{array}$$

This process is continued until the quotient is zero.

$$\begin{array}{r} 0 \\ 12\overline{)4} \\ \underline{0} \\ R_3 = 4_8 = 4 \end{array}$$

Now, if $634_8 = X_{10}$ then

$$X_{10} = R_3 \quad R_2 \quad R_1$$

where

$$R_1 = 2$$

$$R_2 = 1$$

$$R_3 = 4$$

Therefore

$$X_{10} = 412$$

EXAMPLE: Convert $7t0e_{12}$ to decimal.
SOLUTION: Ten expressed in base twelve is t, therefore, write

$$t\overline{)7t0e}$$

This division is carried out in base twelve.

$$\begin{array}{r} 94t \\ t\overline{)7t0e} \\ \underline{76} \\ 40 \\ \underline{34} \\ 8e \\ \underline{84} \\ R_1 = 7_{12} = 7 \end{array}$$

Divide the quotient as follows:

$$\begin{array}{r} e3 \\ t\overline{)94t} \\ \underline{92} \\ 2t \\ \underline{26} \\ R_2 = 4_{12} = 4 \end{array}$$

Then, the quotient e3 is divided by t. That is,

$$\begin{array}{r} 11 \\ t\overline{)e3} \\ \underline{t} \\ 13 \\ \underline{t} \\ R_3 = 5_{12} = 5 \end{array}$$

Further division produces

$$\begin{array}{r} 1 \\ t\overline{)11} \\ \underline{t} \\ R_4 = 3_{12} = 3 \end{array}$$

Then,

$$\begin{array}{r} 0 \\ t\overline{)1} \\ \underline{0} \\ R_5 = 1_{12} = 1 \end{array}$$

Therefore, if $7t0e_{12} = X_{10}$ and

$$X_{10} = R_5 \quad R_4 \quad R_3 \quad R_2 \quad R_1$$

where

$$R_1 = 7$$

$$R_2 = 4$$

$$R_3 = 5$$

$$R_4 = 3$$

$$R_5 = 1$$

then

$$X_{10} = 13547$$

PROBLEMS: Convert the following non-decimal numbers to decimal using each of the three methods discussed.

1. 342_8
2. 431_5
3. $6AC_{16}$

ANSWERS:

1. 226
2. 116
3. 1708

DECIMAL TO NON-DECIMAL

To convert a number from decimal to non-decimal the process of repeated division is used and the remainders indicate the non-decimal number. The divisor is the non-decimal base expressed in base ten and the division process is in base ten.

EXAMPLE: Convert 319 to octal; that is, if

$$319 = X_8, \text{ then } X_8 = ?$$

SOLUTION: Base eight expressed in decimal is 8, therefore, write

$$\begin{array}{r} 39 \\ 8\overline{)319} \\ \underline{24} \\ 79 \\ \underline{72} \\ R_1 = 7 = 7_8 \end{array}$$

and

$$\begin{array}{r} 4 \\ 8\overline{)39} \\ \underline{32} \\ R_2 = 7 = 7_8 \end{array}$$

then

$$\begin{array}{r} 0 \\ 8\overline{)4} \\ \underline{0} \\ R_3 = 4 = 4_8 \end{array}$$

Now, if

$$319 = X_8$$

then

$$X_8 = R_3\ R_2\ R_1$$

In this example

$$R_1 = 7$$

$$R_2 = 7$$

$$R_3 = 4$$

therefore,

$$X_8 = 477_8$$

EXAMPLE: Convert 18 to binary; that is, if

$$18 = X_2, \text{ then } X_2 = ?$$

SOLUTION: Base two in decimal is 2, therefore, write

$$\begin{array}{r} 9 \\ 2\overline{)18} \\ \underline{18} \\ R_1 = 0 = 0_2 \end{array}$$

and

$$\begin{array}{r} 4 \\ 2\overline{)9} \\ \underline{8} \\ R_2 = 1 = 1_2 \end{array}$$

and

$$\begin{array}{r} 2 \\ 2\overline{)4} \\ \underline{4} \\ R_3 = 0 = 0_2 \end{array}$$

and

$$\begin{array}{r} 1 \\ 2\overline{)2} \\ \underline{2} \\ R_4 = 0 = 0_2 \end{array}$$

then

$$\begin{array}{r} 0 \\ 2\overline{)1} \\ \underline{0} \\ R_5 = 1 = 1_2 \end{array}$$

therefore, if $18 = X_2$ and

$$X_2 = R_5\ R_4\ R_3\ R_2\ R_1$$

where

$$R_1 = 0$$

$$R_2 = 1$$

$$R_3 = 0$$

$$R_4 = 0$$

$$R_5 = 1$$

then

$$X_2 = 10010_2$$

EXAMPLE: Convert 632 to duodecimal. That is, if

$$632 = X_{12}, \text{ then } X_{12} = ?$$

SOLUTION: Base twelve expressed in base ten is 12. Write

$$\begin{array}{r} 52 \\ 12\overline{)632} \\ \underline{60} \\ 32 \\ \underline{24} \\ R_1 = 8 = 8_{12} \end{array}$$

and

$$\begin{array}{r} 4 \\ 12\overline{)52} \\ \underline{48} \\ R_2 = 4 = 4_{12} \end{array}$$

and

$$\begin{array}{r} 0 \\ 12\overline{)4} \\ \underline{0} \\ R_3 = 4 = 4_{12} \end{array}$$

If

$$632 = X_{12}$$

and

$$X_{12} = R_3 \quad R_2 \quad R_1$$

then

$$X_{12} = 448_{12}$$

EXAMPLE: Convert 128 to duodecimal.
SOLUTION: Write

$$\begin{array}{r} 10 \\ 12\overline{)128} \\ \underline{12} \\ R_1 = 8 = 8_{12} \end{array}$$

and

$$\begin{array}{r} 0 \\ 12\overline{)10} \\ \underline{00} \\ R_2 = 10 = t_{12} \end{array}$$

therefore,

$$128 = t8_{12}$$

The format for the repeated division process may be simplified in cases where the actual division is simple. This is shown in the following example.

EXAMPLE: Convert 18 to binary.

SOLUTION: Carry out the repeated division indicating the remainder to the right of the division; that is,

$$\begin{array}{ll} & \text{Remainder} \\ 2\overline{)18} & \\ 2\overline{)9} & 0 \uparrow \\ 2\overline{)4} & 1 \\ 2\overline{)2} & 0 \\ 2\overline{)1} & 0 \\ 0 & 1 \end{array}$$

Now read the remainder from the bottom to the top to find the binary equivalent of the decimal number. In this case the binary number is 10010_2.

PROBLEMS: Convert the following decimal numbers to the base indicated.

1. 27 to base two.

2. 123 to base five.

3. 467 to base twelve.

4. 996 to base sixteen.

ANSWERS:

1. 11011_2
2. 443_5
3. $32e_{12}$
4. $3E4_{16}$

NON-DECIMAL TO NON-DECIMAL

We will consider three approaches to the non-decimal to non-decimal conversions. One method will be through base ten and the other two methods will be direct.

When going through base ten, the polynomial form is used along with repeated division.

EXAMPLE: Convert 2143_5 to base eight.

SOLUTION: Convert 2143_5 to base ten by writing this number in polynomial form. That is,

$$\begin{aligned} 2143_5 &= 2(5)^3 + 1(5)^2 + 4(5)^1 + 3(5)^0 \\ &= 2(125) + 1(25) + 4(5) + 3(1) \\ &= 250 + 25 + 20 + 3 \\ &= 298 \end{aligned}$$

We now convert 298 from base ten to base eight by repeated division.. That is, write

	Remainder
8/298	
8/37	2
8/4	5
0	4

therefore,

$$2143_5 = 452_8$$

EXAMPLE: Convert 10110_2 to base twelve.

SOLUTION: Write (polynomial form)

$$\begin{aligned} 10110_2 &= 1(2)^4 + 0(2)^3 + 1(2)^2 + 1(2)^1 + 0(2)^0 \\ &= 16 + 0 + 4 + 2 + 0 \\ &= 22 \end{aligned}$$

Then (repeated division),

	Remainder
12/22	
12/1	t
0	1

therefore,

$$10110_2 = 1t_{12}$$

EXAMPLE: Convert $3C7_{16}$ to base five.

SOLUTION: Write (polynomial form)

$$\begin{aligned} 3C7_{16} &= 3(16)^2 + C(16)^1 + 7(16)^0 \\ &= 768 + 192 + 7 \\ &= 967 \end{aligned}$$

Then (repeated division),

	Remainder
5/967	
5/193	2
5/38	3
5/7	3
5/1	2
0	1

therefore,

$$3C7_{16} = 12332_5$$

The second method of converting a non-decimal number to a non-decimal number is by division. The division is carried out by dividing by the base wanted, performing the calculation in the base given.

EXAMPLE: Convert 2143_5 to base eight.

SOLUTION: The base given is five and the base wanted is eight. Therefore, express eight in base five, obtaining 13_5. We carry out the division by 13_5 in base five as follows:

$$\begin{array}{r} 122 \\ 13\overline{)2143} \\ \underline{13} \\ 34 \\ \underline{31} \\ 33 \\ \underline{31} \\ R_1 = 2_5 = 2_8 \end{array}$$

then

$$\begin{array}{r} 4 \\ 13\overline{)122} \\ 112 \\ \hline R_2 = 10_5 = 5_8 \end{array}$$

and

$$\begin{array}{r} 0 \\ 13\overline{)4} \\ 0 \\ \hline R_3 = 4_5 = 4_8 \end{array}$$

and

$$X_8 = R_3 \quad R_2 \quad R_1$$

then

$$\begin{aligned} 2143_5 &= X_8 \\ &= R_3 \quad R_2 \quad R_1 \\ &= 452_8 \end{aligned}$$

EXAMPLE: Convert $7e6_{12}$ to base five.

SOLUTION: The base given is twelve and the base wanted is five. Therefore, five in base twelve is 5_{12}. The division is carried out in base twelve. Write

$$\begin{array}{r} 171 \\ 5\overline{)7e6} \\ 5 \\ \hline 2e \\ 2e \\ \hline 6 \\ 5 \\ \hline R_1 = 1_{12} = 1_5 \end{array}$$

then

$$\begin{array}{r} 39 \\ 5\overline{)171} \\ 13 \\ \hline 41 \\ 39 \\ \hline R_2 = 4_{12} = 4_5 \end{array}$$

and

$$\begin{array}{r} 9 \\ 5\overline{)39} \\ 39 \\ \hline R_3 = 0_{12} = 0_5 \end{array}$$

and

$$\begin{array}{r} 1 \\ 5\overline{)9} \\ 5 \\ \hline R_4 = 4_{12} = 4_5 \end{array}$$

then

$$\begin{array}{r} 0 \\ 5\overline{)1} \\ 0 \\ \hline R_4 = 1_{12} = 1_5 \end{array}$$

therefore,

$$7e6_{12} = 14041_5$$

PROBLEMS: Convert the following without going through base ten.

1. 342_5 to base two.
2. $t73_{12}$ to base eight.
3. $A62_{16}$ to base five.

ANSWERS:

1. 1100001_2
2. 2767_8
3. 41113_5

The last method we will discuss is called the explosion method. It consists of the following rules:

1. Perform all arithmetic operations in the desired base.
2. Express the base of the original number in terms of the base of the desired number.
3. Multiply the number obtained in step 2 by the leftmost digit and add the product to the next digit on the right of the original number.

(NOTE: It may be necessary to convert each digit of the original number to an expression conforming to the desired base.)

4. Repeat step 3 as many times as there are digits. The final sum is the answer.

EXAMPLE: Convert 2143_5 to base eight.
SOLUTION: Write

five in base eight = 5_8 (Rule 2)

$$\begin{array}{cccc} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & & & \\ 12_8 & & & \end{array}$$

(Rule 3)

then

$$\begin{array}{cccc} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & & \\ 12_8 & 13_8 & & \\ & \underline{\times\ 5_8} & & \\ & 67_8 & & \end{array}$$

and

$$\begin{array}{cccc} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & \underline{+\ 67_8} & \\ 12_8 & 13_8 & 73_8 & \\ & \underline{\times\ 5_8} & \underline{\times\ 5_8} & \\ & 67_8 & 447_8 & \end{array}$$

then

$$\begin{array}{cccc} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & \underline{+\ 67_8} & \underline{+\ 447_8} \\ 12_8 & 13_8 & 73_8 & 452_8 \\ & \underline{\times\ 5_8} & \underline{\times\ 5_8} & \\ & 67_8 & 447_8 & \end{array}$$

Therefore,

$$2143_5 = 452_8$$

EXAMPLE: Convert 452_8 to base five.
SOLUTION: Eight expressed in base five is 13_5. 452 expressed in base five, digit by digit, is 4 10 2. Then,

$$\begin{array}{ccc} 4 & 10 & 2 \\ \underline{\times\ 13_5} & \underline{+\ 112} & \underline{+\ 2141} \\ 22 & 122 & 2143_5 \\ \underline{4} & \underline{\times\ 13} & \\ 112 & 421 & \\ & \underline{122} & \\ & 2141 & \end{array}$$

Therefore,

$$452_8 = 2143_5$$

EXAMPLE: Convert $t62_{12}$ to base five.
SOLUTION: Twelve expressed in base five is 22_5. t62 expressed in base five, digit by digit, is 20 11 2. Then,

$$\begin{array}{ccc} 20 & 11 & 2 \\ \underline{\times\ 22_5} & \underline{+\ 440} & \underline{+\ 22022} \\ 40 & 1001 & 22024_5 \\ \underline{40} & \underline{\times\ 22} & \\ 440 & 2002 & \\ & \underline{2002} & \\ & 22022 & \end{array}$$

therefore,

$$t62_{12} = 22024_5$$

PROBLEMS: Convert the following to the base indicated.

1. 32_5 to base two.
2. 347_8 to base twelve.
3. $te3_{12}$ to base five.

ANSWERS:

1. 10001_2

2. 173_{12}

3. 22300_5

SPECIAL CASES OF CONVERSIONS

Changing from base two to base eight is accomplished rather easily because eight is a power of two. That is, 8 equals 2^3. We need only group our base two number in groups of three digits (the power of the original which gives the new base) and use each group of three digits as a single place value of base eight.

EXAMPLE: Convert 1011001_2 to base eight.

SOLUTION: Group 1011001_2 in groups of three starting at the right. That is,

$$001 \quad 011 \quad 001 \;_2$$

then write each group of three digits in base eight.

$$1 \quad 3 \quad 1 \;_8$$

Verification may be made by writing

$$1011001_2 = 89$$

and

$$131_8 = 89$$

therefore,

$$1011001_2 = 131_8$$

EXAMPLE: Convert 1001101_2 to base sixteen.

SOLUTION: Sixteen is the fourth power of two so we use groups of four digits. Write

$$0100 \quad 1101 \;_2$$

Then write each group of digits in base sixteen.

$$4 \quad D \;_{16}$$

EXAMPLE: Convert 11011011111_2 to base sixteen.

SOLUTION: Write

$$0110 \quad 1101 \quad 1111 \;_2$$

$$= \; 6 \quad D \quad F \;_{16}$$

To reverse this process, that is, to convert from base eight to base two, we use three digits in base two to express each digit in base eight. This is because two is the third root of eight.

EXAMPLE: Convert 132_8 to base two.

SOLUTION: Write

$$1 \quad 3 \quad 2 \;_8$$

Then write each base eight digit in base two using three digits. That is,

$$1 \quad 3 \quad 2 \;_8$$

$$= \; 001 \quad 011 \quad 010 \;_2$$

EXAMPLE: Convert $6A7_{16}$ to base two.

SOLUTION: Two is the fourth root of sixteen; therefore, we express each base sixteen digit in base two using four digits. That is,

$$6 \quad A \quad 7 \;_{16}$$

$$= \; 0110 \quad 1010 \quad 0111 \;_2$$

This process may be used when one base is a power or root of the other base.

PROBLEMS: Convert the following to the base indicated.

1. 11011101_2 to base eight.

2. 4762_8 to base two.

3. $9B7_{16}$ to base two.

4. 111110111_2 to base sixteen.

ANSWERS:

1. 335_8

2. 100111110010_2

3. 100110110111_2

4. $1F7_{16}$

DECIMAL TO BINARY CODED DECIMAL

Although the Binary Coded Decimal (BCD) is not truly a number system, we will discuss this code because it is computer related as some of the bases are.

This code, sometimes called the 8421 code, makes use of groups of binary symbols to represent a decimal number. In the decimal system there are only ten symbols; therefore, only ten groups of binary bits (symbols) must be remembered. Each decimal digit is represented by a group of four binary bits. The ten groups to remember are as follows:

Decimal Symbol	Binary Coded Decimal (BCD)
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

To express a decimal number as a BCD we use a binary group for each decimal symbol we have; that is,

Decimal	7	2
BCD	0111	0010

Notice that for every decimal digit we must have one group of binary bits. Thus,

Decimal	3	8	1
BCD	0011	1000	0001

The separation of the BCD groups is shown for ease of reading and does not necessarily need to be written as shown. The number 381 could be written as 001110000001. One advantage of the BCD over true binary is ease of determining the decimal value. This is shown as follows:

Decimal	9	3	4
BCD	1001	0011	0100

The number 934 in true binary is 1110100110_2. This in polynomial form is

$$1(2)^9 + 1(2)^8 + 1(2)^7 + 0(2)^6 + 1(2)^5 + 0(2)^4 + 0(2)^3 + 1(2)^2 + 1(2)^1 + 0(2)^0$$

$$= 512 + 256 + 128 + 0 + 32 + 0 + 0 + 4 + 2 + 0$$

$$= 934$$

In order to change a decimal to BCD we need only write one group of binary bits to represent each decimal digit; that is,

Decimal 7203

is

BCD 0111 0010 0000 0011

To convert a BCD to decimal we group the binary bits in groups of four, from the right, and then write the decimal digit represented by each group. Thus,

BCD	0100100110011000			
= BCD	0100	1001	1001	1000
= Decimal	4	9	9	8
= 4998				

The ease of converting from decimal to BCD and from BCD back to decimal should be apparent from the following problems.

PROBLEMS: Convert the following decimal numbers to BCD.

1. 6
2. 31
3. 764
4. 3098

ANSWERS:

1. 0110
2. 0011 0001 or 00110001

3. 0111 0110 0100 or 011101100100

4. 0011 0000 1001 1000 or 0011000010011000

PROBLEMS: Convert the following BCD's to decimals.

1. 0010

2. 10011000

3. 011000110111

4. 0101000001111000

ANSWERS:

1. 2

2. 98

3. 637

4. 5078

The comparative ease of conversion in BCD is related to the difficulty of conversion in true binary by the following problems.

PROBLEMS: Convert as follows:

1. 438 to binary

2. 100101101_2 to decimal

ANSWERS:

1. 110110110_2

2. 301

One serious disadvantage of the BCD is that this code cannot provide a "decimal" carry. The following examples are given to show this.

EXAMPLE: Add the following:

Decimal		BCD
5	=	0101
+ 3	=	+ 0011
8	=	1000

Notice that we did not have a carry in the decimal addition and the answer in BCD is equal to the answer in decimal. The BCD is in correct notation and does exist.

EXAMPLE: Add the following:

Decimal		BCD
8	=	1000
+ 5	=	+ 0101
13	=	1101

Notice that the BCD symbol is the true binary representation of 13 but 1101 does not exist in BCD. The correct BCD answer for 13 is 0001 0011. When a carry is made in decimal the BCD system cannot indicate the correct answer in BCD form.

EXCESS THREE CODE

The excess three code is used to eliminate the inability of the decimal carry. It is really a modification of the BCD so that a carry can be made.

To change a BCD symbol to excess three add three to the BCD; that is,

BCD	1000
	+ 0011
excess three	1011

The excess three number 1011 is 8 in decimal. The following shows the correspondence between decimal, BCD, and excess three code.

Decimal	BCD	Excess Three
0	0000	0011
1	0001	0100
2	0010	0101
3	0011	0110
4	0100	0111
5	0101	1000
6	0110	1001
7	0111	1010
8	1000	1011
9	1001	1100

As previously stated, the excess three code will provide the capability of the decimal carry. The following is given for explanation.

EXAMPLE: Add 6 and 3 in excess three.
SOLUTION: Write

Decimal				
6	=	0011	0011	1001
+ 3	=	0011	0011	0110
9				

Notice that in the right-hand groups the six and three are given. In the other groups a zero (0011) is indicated.

Then,

0011	0011	1001	(excess three)
+ 0011	0011	0110	(excess three)
0110	0110	1111	(excess six)

Our answer is in excess six; therefore, we must subtract three from each group in order to return our answer to excess three; that is,

0110	0110	1111	
- 0011	0011	0011	
0011	0011	1100	(excess three)
0	0	9	in decimal

When a carry is developed in any group, the following procedure is used.

EXAMPLE: Add 9 and 3 in excess three.
SOLUTION: Write

Decimal	Excess Three			
9	0011	0011	1100	(excess three)
		1		
+ 3	0011	0011	0110	(excess three)
12	0110	0111	0010	(excess six)

NOTE: Since the right-hand group created a carry, as shown, three must be ADDED instead of subtracted in order to place this group into excess three. The other groups follow the previous example; that is,

0110	0111	0010	(excess six)
- 0011	-0011	+0011	
0011	0100	0101	(excess three)
0	1	2	decimal

PROBLEMS: Add the following in excess three.

1. 5 and 3.
2. 9 and 8.
3. 22 and 56.
4. 58 and 77.

ANSWERS:

1. 0011 0011 1011
2. 0011 0100 1010
3. 0011 1010 1011
4. 0100 0110 1000

A further advantage of the excess three code is the ease with which the nines complement of a number indicated in excess three may be found. That is, the nines complement of seven, indicated in excess three as 1010, is found by inverting each digit in 1010 to read 0101. This 0101, in excess three represents decimal two which is the nines complement of seven.

The following shows the nines complement of the decimal digits.

Decimal	Excess Three	Excess Three Nines Complement	Decimal Nines Complement
0	0011	1100	9
1	0100	1011	8
2	0101	1010	7
3	0110	1001	6
4	0111	1000	5
5	1000	0111	4
6	1001	0110	3
7	1010	0101	2
8	1011	0100	1
9	1100	0011	0